

On supersymmetric AdS_6 solutions in 10 and 11 dimensions

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Abstract

We prove a non-existence theorem for smooth, supersymmetric, warped AdS_6 solutions with connected, compact without boundary internal space in $D = 11$ and (massive) IIA supergravities. In IIB supergravity we show that if such AdS_6 solutions exist, then the NSNS and RR 3-form fluxes must be linearly independent and certain spinor bi-linears must be appropriately restricted. Moreover we demonstrate that the internal space admits an $\mathfrak{so}(3)$ action which leaves all the fields invariant and for smooth solutions the principal orbits must have co-dimension two. We also describe the topology and geometry of internal spaces that admit such a $\mathfrak{so}(3)$ action and show that there are no solutions for which the internal space has topology $F \times S^2$, where F is an oriented surface.

1 Introduction

AdS spaces have found widespread applications first as compactifications of supergravity theories and more recently as a tool to explore superconformal field theories in the context of the AdS/CFT correspondence, for reviews see [1, 2]. As AdS/CFT provides a correspondence between AdS_n backgrounds of 10 and 11-dimensional supergravity theories with conformal field theories in $(n - 1)$ -dimensions, properties of superconformal theories can be investigated in the context of supergravity theories. This has given a new impetus to understanding the AdS backgrounds that preserve a fraction of the spacetime supersymmetry.

In this context, and in particular for AdS_6 backgrounds, several solutions have been found in [3, 4, 5, 6, 7, 8, 9] and explored in the context of AdS_6/CFT_5 , see eg [10, 11, 12, 13, 14]. Furthermore it has been shown in [15, 16, 17] that AdS_6 backgrounds preserve either 16 or 32 supersymmetries in all 10- and 11-dimensional supergravity theories. Moreover it is known for sometime that these theories do not admit maximally supersymmetric backgrounds [18] which are strictly locally isometric to AdS_6 . Some additional non-existence results have been established in [6, 19] for smooth AdS_6 solutions preserving 16 supersymmetries in 11-dimensional and IIA supergravities under the assumption that the Killing spinors factorize into Killing spinors of AdS and Killing spinors on the internal space and some additional restrictions on the internal spaces.

It has been demonstrated in [20] that the Killing spinors of AdS backgrounds do not factorize into Killing spinors of AdS and Killing spinors on the internal space, and so requiring factorization is an additional assumption. Related to this some care is required in establishing no-go theorems for AdS backgrounds as AdS_n spaces can be written as warped products of AdS_k , $k < n$ and so AdS_n backgrounds can be re-interpreted as warped AdS_k solutions [21]-[24]. For example the AdS_7 maximally supersymmetric background of 11-dimensional supergravity can be re-interpreted as a warped maximally supersymmetric AdS_6 solution. To exclude such a scenario, we put some global assumptions on the internal spaces of AdS_6 backgrounds that we shall describe below.

In this paper, we shall prove a non-existence theorem for smooth warped AdS_6 backgrounds in 11-dimensional and (massive) IIA imposing only as assumptions¹ that the internal space is closed², ie it is compact and without boundary. As this theorem for maximally supersymmetric backgrounds has already been demonstrated, the main focus is to establish the result for AdS_6 solutions preserving 16 supersymmetries.

Furthermore we shall demonstrate some non-existence results for smooth closed AdS_6 backgrounds in IIB supergravity provided some additional assumptions are made. In turn these assumptions can be viewed as necessary conditions for the existence of IIB AdS_6 solutions. In particular, we demonstrate that if smooth closed AdS_6 IIB backgrounds exist, the NSNS and RR 3-form fluxes must be linearly independent. This rules out the existence of smooth closed solutions with only NSNS or RR 3-form fluxes and all their $SL(2, \mathbb{R})$ duals. We also find that this linear independence condition on the 3-form fluxes is met provided some spinor bilinears are appropriately restricted. In particular, a certain

¹In particular, we do not assume that the Killing spinors are factorized as described above.

²For simplicity, we shall refer to smooth AdS backgrounds with closed internal space as “smooth closed AdS backgrounds”.

(twisted) scalar bilinear must vanish, another (twisted) scalar bilinear must be somewhere vanishing on the internal space and a 2-form bilinear must have rank at most two. The full set of conditions is summarized in section 4.4. Another necessary condition is that closed AdS_6 IIB backgrounds must always have active scalars.

Next we show that the internal spaces of all AdS_6 IIB backgrounds admit a non-trivial³ $\mathfrak{so}(3)$ action with leaves all the fields invariant. If the infinitesimal $\mathfrak{so}(3)$ action can be integrated to an effective $SU(2)$ or $SO(3)$ action and AdS_6 IIB backgrounds are smooth and closed, then the principal orbits must be of co-dimension 2. This rules out all solutions for which $\mathfrak{so}(3)$ acts on the internal space with co-dimension 1 principal orbits. The diffeomorphic type of internal manifolds in the oriented case can be specified by utilizing the classification results of [25]. These include the spin manifolds

$$S^4, \quad p(S^1 \times S^3) \# q(S^1 \times \mathbb{R}P^3), \quad M(F), \quad (1.1)$$

where $M(F)$ is the unique spin oriented 2-sphere bundle over a surface F , see also appendix C. In particular if F is oriented, then $M(F) = F \times S^2$. Next we demonstrate with a partial integration argument that there are no smooth closed AdS_6 solutions that have internal spaces with topology $F \times S^2$.

The methodology that will be followed has been developed in [15, 16, 17] for investigating all AdS backgrounds. In particular, it is assumed that the metric and fluxes are invariant under the isometries of the AdS space but otherwise there are no additional assumptions like an ansatz on the form of the Killing spinors. Then the supergravity KSEs are integrated along the AdS space which gives the dependence of the Killing spinor on the AdS coordinates as well as a set of KSEs along the internal space. These are associated with the gravitino and other algebraic KSEs of the original supergravity theory, and in addition there is an algebraic Killing spinor equation which arises as an integrability condition associated with the solution of the KSEs along the AdS . The existence of smooth closed solutions is then explored by applying techniques, like that of the Hopf maximum principle, and taking into account the assumptions made on the topology of the internal space. It turns out that this methodology is sufficient to prove the non-existence of supersymmetric AdS_6 backgrounds in 11-dimensional and (massive) IIA supergravities. However in IIB, we have not been able to establish such a result. Instead, we have given some necessary conditions for the existence of smooth closed AdS_6 solutions, and specified their diffeomorphic type using classification results of [25] for 4-dimensional manifolds admitting an effective $\mathfrak{so}(3)$ action.

This paper has been organized as follows. In sections two and three, we establish the non-existence of smooth closed AdS_6 backgrounds for 11-dimensional and (massive) IIA supergravities, respectively. In section four, we investigate the existence of smooth closed AdS_6 solutions in IIB supergravity. In appendix A, we investigate the isometries of the internal space and state our conventions. In appendix B, we present some Fierz identities that have been used in our derivations. In appendix C, we summarize the results of [25] on the structure of 4-manifolds admitting a non-abelian group action and in appendix D, we present various formulae for the spinor bilinears of IIB AdS_6 backgrounds.

³This means that the associated Killing vector fields do not vanish everywhere on the internal space.

2 Warped AdS_6 backgrounds in D=11

We begin by briefly summarizing the general structure of warped AdS_6 solutions in 11-dimensional supergravity, as determined in [15], whose conventions we shall follow throughout this section. The metric and 4-form which are invariant under the isometries of AdS_6 are given by

$$\begin{aligned} ds^2 &= 2du(dr + rh) + A^2(dz^2 + e^{2z/\ell} \sum_{a=1}^3 (dx^a)^2) + ds^2(M^5) , \\ F &= X , \end{aligned} \quad (2.1)$$

where $ds^2(M^5)$ is the metric on the internal space M^5 , and we have written the solution as a near-horizon geometry [26], with

$$h = -\frac{2}{\ell}dz - 2A^{-1}dA . \quad (2.2)$$

The coordinates (u, r, z, x^1, x^2, x^3) are those of the AdS_6 space, A is the warp factor which is a function on M^5 , and X is a closed 4-form on M^5 . The metric $ds^2(M)$, A and X depend only on the coordinates of M^5 , ℓ is the radius of AdS_6 .

The 11-dimensional Einstein equation implies that the warp factor satisfies the equation

$$D^k \partial_k \log A = -\frac{5}{\ell^2} A^{-2} - 6 \partial^k \log A \partial_k \log A + \frac{1}{144} X^2 , \quad (2.3)$$

where D is the Levi-Civita connection on M^5 . The remaining components of the Einstein and gauge field equations are listed in [15], however we shall only require (2.3) for the analysis that follows. In particular, (2.3) implies that A is everywhere non-vanishing on M^5 , on assuming that M^5 is connected and all fields are smooth.

2.1 The Killing spinors

The KSEs of 11-dimensional supergravity can be solved along AdS_6 yielding

$$\begin{aligned} \epsilon &= \sigma_+ - \ell^{-1} \sum_{a=1}^3 x^a \Gamma_{az} \tau_+ + e^{-\frac{z}{\ell}} \tau_+ + \sigma_- + e^{\frac{z}{\ell}} (\tau_- - \ell^{-1} \sum_{a=1}^3 x^a \Gamma_{az} \sigma_-) \\ &\quad - \ell^{-1} u A^{-1} \Gamma_{+z} \sigma_- - \ell^{-1} r A^{-1} e^{-\frac{z}{\ell}} \Gamma_{-z} \tau_+ , \end{aligned} \quad (2.4)$$

where the spinors σ_{\pm} and τ_{\pm} are Majorana $Spin(10,1)$ spinors that depend only on the coordinates y^I of M^5 and satisfy the light-cone projections

$$\Gamma_{\pm} \sigma_{\pm} = 0 , \quad \Gamma_{\pm} \tau_{\pm} = 0 . \quad (2.5)$$

The gamma matrices have been adapted to the spacetime frame

$$\begin{aligned} \mathbf{e}^+ &= du , & \mathbf{e}^- &= dr + rh , & \mathbf{e}^z &= Adz , & \mathbf{e}^a &= A e^{z/\ell} dx^a , \\ \mathbf{e}^i &= e^i_I dy^I , \end{aligned} \quad (2.6)$$

where $ds^2(M^5) = \delta_{ij}e^i_I e^j_J dy^I dy^J$. The expression for the Killing spinor ϵ (2.4) is derived after integrating the KSEs along AdS_6 . In particular notice that we do not assume that the Killing spinor ϵ factorizes as a Killing spinor on AdS_6 and a Killing spinor on M^5 which is an additional assumption on the form of the Killing spinors [15, 20].

The remaining independent Killing spinor equations (KSEs) are

$$D_i^{(\pm)} \sigma_{\pm} = 0, \quad D_i^{(\pm)} \tau_{\pm} = 0, \quad (2.7)$$

and

$$\Xi^{(\pm)} \sigma_{\pm} = 0, \quad \Xi^{(\mp)} \tau_{\pm} = 0, \quad (2.8)$$

where

$$\begin{aligned} D_i^{(\pm)} &= D_i \pm \frac{1}{2} \partial_i \log A - \frac{1}{288} \Gamma X_i + \frac{1}{36} X_i, \\ \Xi^{(\pm)} &= -\frac{1}{2} \Gamma_z \Gamma^i \partial_i \log A \mp \frac{1}{2\ell} A^{-1} + \frac{1}{288} \Gamma_z X. \end{aligned} \quad (2.9)$$

The (2.7) KSEs are a suitable restriction of the gravitino KSE of 11-dimensional supergravity on M^5 while the (2.8) conditions arise during the integration process of the KSEs along AdS_6 . Notice that the algebraic KSEs (2.8) imply that σ_+ and τ_+ cannot be linearly dependent. In fact, we shall later demonstrate that they must be orthogonal. For our Clifford algebra conventions see [15].

2.2 Counting the Killing Spinors

The counting of supersymmetries of warped AdS_6 backgrounds will be given in 11-dimensional supergravity. A similar counting applies to (massive) IIA and IIB supergravities and so it will not be repeated below for these theories.

To begin, note that if σ_+ is a solution of the σ_+ KSEs, then so is $\Gamma_{ab}\sigma_+$ for $a, b = 1, 2, 3$. Furthermore, $\tau_+ = \Gamma_{za}\sigma_+$ are also solutions to the τ_+ KSEs. The eight spinors $\sigma_+, \Gamma_{ab}\sigma_+, \Gamma_{za}\sigma_+, \Gamma_{123z}\sigma_+$ are linearly independent.

The spinors σ_-, τ_- can also be constructed from σ_+ and τ_+ . This is because if σ_+, τ_+ is a solution, then so is $\sigma_- = A\Gamma_{-z}\sigma_+, \tau_- = A\Gamma_{-z}\tau_+$ and conversely, if σ_-, τ_- is a solution, then so is $\sigma_+ = A^{-1}\Gamma_{+z}\sigma_-, \tau_+ = A^{-1}\Gamma_{+z}\tau_-$. Thus all Killing spinors of AdS_6 backgrounds are generated by the σ_+ Killing spinors.

As a result the number of Killing spinors of AdS_6 backgrounds is a multiple of 16. Thus if there are AdS_6 solutions of supergravity theories, they will either preserve 16 or 32 supersymmetries. It has been shown sometime ago that 11-dimensional, (massive) IIA and IIB supergravities do not admit maximally supersymmetric AdS_6 solutions [18]. As a result, it remains to investigate the AdS_6 backgrounds that preserve 16 supersymmetries.

2.3 Proof of the main theorem in $D = 11$ supergravity

2.3.1 Orthogonality of τ_+ and σ_+ spinors

Before we proceed with the proof of the main theorem, we shall first establish the orthogonality of τ_+ and σ_+ Killing spinors. It will be convenient to define

$$W = \tilde{\star} X, \quad (2.10)$$

where $\tilde{\star}$ denotes the Hodge dual on M^5 , so W is a 1-form. To proceed, we set

$$\Lambda = \sigma_+ + \tau_+ , \quad (2.11)$$

and using (2.7), we find

$$D_+ \parallel \Lambda \parallel^2 = -A^{-1} D_i A \parallel \Lambda \parallel^2 + \frac{1}{6} W_i \langle \Lambda, \Gamma_{(4)} \Lambda \rangle , \quad (2.12)$$

where $\Gamma_{(4)}$ denotes the product of the gamma matrices in the 4 directions of AdS_6 spanned by \mathbf{e}^z and \mathbf{e}^a , with the convention that

$$\Gamma_{(4)} \Gamma_{i_1 i_2 i_3 i_4 i_5} \phi_{\pm} = \pm \epsilon_{i_1 i_2 i_3 i_4 i_5} \phi_{\pm} . \quad (2.13)$$

Using the algebraic KSE (2.8), (2.12) can be rewritten as

$$D_i \parallel \Lambda \parallel^2 = -\frac{2}{\ell} A^{-1} \langle \sigma_+, \Gamma_i \Gamma_z \tau_+ \rangle \quad (2.14)$$

and the gravitino KSE (2.7) implies that

$$D^i \langle \sigma_+, \Gamma_i \Gamma_z \tau_+ \rangle = -\langle \sigma_+, \Gamma_i \Gamma_z A^{-1} D^i A \tau_+ \rangle . \quad (2.15)$$

On taking the divergence of (2.14), and utilizing (2.15), we find

$$D^i D_i \parallel \Lambda \parallel^2 + 2A^{-1} D^i A D_i \parallel \Lambda \parallel^2 = 0 . \quad (2.16)$$

A maximum principle argument then implies that

$$\parallel \Lambda \parallel^2 = \text{const} . \quad (2.17)$$

Note that for the application of the maximum principle it is sufficient to assume that the backgrounds are smooth and M^5 is closed. As $\parallel \Lambda \parallel^2$ is constant, (2.12) and (2.14) imply

$$-A^{-1} D_i A \parallel \Lambda \parallel^2 + \frac{1}{6} W_i \langle \Lambda, \Gamma_{(4)} \Lambda \rangle = 0 , \quad (2.18)$$

and

$$\langle \sigma_+, \Gamma_i \Gamma_z \tau_+ \rangle = 0 , \quad (2.19)$$

respectively.

Next taking inner products of the algebraic KSE (2.8) acting on σ_+ and τ_+ with τ_+ and σ_+ respectively, one finds, on using (2.19), that

$$\begin{aligned} -\frac{1}{2\ell} \langle \tau_+, \sigma_+ \rangle - \frac{1}{12} \langle \tau_+, W \Gamma_z \Gamma_{(4)} \sigma_+ \rangle &= 0 , \\ \frac{1}{2\ell} \langle \sigma_+, \tau_+ \rangle - \frac{1}{12} \langle \sigma_+, W \Gamma_z \Gamma_{(4)} \tau_+ \rangle &= 0 . \end{aligned} \quad (2.20)$$

Subtracting these expressions, we deduce that

$$\langle \sigma_+, \tau_+ \rangle = 0 . \quad (2.21)$$

This establishes the orthogonality of σ_+ and τ_+ Killing spinors.

2.3.2 A non-existence theorem

Next, note that as a consequence of how the σ_+ , τ_+ spinors are generated from each other as described in the previous subsection, it follows that $\Gamma_{(4)}\sigma_+ = \tau'_+$, and hence (2.21) implies that

$$\langle \sigma_+, \Gamma_{(4)}\sigma_+ \rangle = 0 . \quad (2.22)$$

So, on substituting $\Lambda = \sigma_+$ into (2.18), we find

$$dA = 0 , \quad (2.23)$$

and so the warp factor is constant. The gravitino KSE (2.7) also implies that

$$D_i \langle \sigma_+, \Gamma_{(4)}\sigma_+ \rangle = \frac{1}{6} W_i \parallel \sigma_+ \parallel^2 , \quad (2.24)$$

and hence this expression together with (2.22) also give that

$$W = 0 , \quad (2.25)$$

and hence $X = 0$. So, we have proven that for AdS_6 solutions, one has

$$dA = 0, \quad X = 0 . \quad (2.26)$$

However, as A is constant and $X = 0$, the field equation (2.3) for the warp factor A does not admit a solution. This proves the theorem.

To summarize, combining the result⁴ of [18] with the proof described above, one concludes that *there are no smooth closed warped AdS_6 solutions of 11-dimensional supergravity*.

One question that arises is whether some of the assumptions we have made can be lifted. First our result can be generalized somewhat. For this notice that we need the smoothness assumption as well as the restrictions on the internal space M^5 to demonstrate that the only solution of (2.16) is that Λ is of constant length (2.17). Thus we can replace all these assumptions with (2.17).

The theorem is not valid if one removes all conditions on the fields and the internal space. This is because locally AdS_{n+1} spaces can be written as warped products of AdS_n spaces⁵. As a result the $AdS_7 \times S^4$ maximally supersymmetric solution of 11-dimensional supergravity can be written locally as $AdS_6 \times_w (\mathbb{R} \times S^4)$ and so it can be interpreted as an AdS_6 solution. This demonstrates that 11-dimensional supergravity admits maximally supersymmetric AdS_6 solutions. However such solutions are not a contradiction to our theorem as their internal space is not compact, see also [20].

⁴For the classification of maximally supersymmetric solutions in [18] there is no need to impose smoothness and compactness conditions. The proof works in general and the backgrounds are classified up to a local isometry.

⁵This was established in [21] for AdS_2 and AdS_3 spaces explored further in [22]-[24] and used in the context of supersymmetric AdS backgrounds in [20].

3 Warped AdS_6 backgrounds in IIA supergravity

The non-vanishing fields of (massive) IIA supergravity for warped $AdS_6 \times_w M^4$ backgrounds in the conventions of [17] are

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + A^2(dz^2 + e^{2z/\ell} \sum_{a=1}^3 (dx^a)^2) + ds^2(M^4) , \\ G &= G , \quad H = H , \quad F = F , \quad \Phi = \Phi , \quad S = S , \end{aligned} \quad (3.1)$$

where A is the warp factor, Φ is the dilaton, S is related to the cosmological constant, F and G are the 2-form and 4-form R-R field strengths correspondingly, and H is a 3-form the NS-NS 3-form field strength. A , Φ and S are functions on the internal space M^4 , while F , G and H are 2-, 4- and 3-forms on M^4 ; all of them depend only on the coordinates of M^4 as well as the metric of the internal space $ds^2(M^4)$. We have also introduced the frame $(\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^z, \mathbf{e}^a, \mathbf{e}^i)$ as in (2.6) and ℓ is the radius of AdS_6 . It will be convenient to define

$$H_{ijk} = \epsilon_{ijk}{}^\ell W_\ell, \quad G_{ijkl} = X \epsilon_{ijkl} \quad (3.2)$$

where W is now a 1-form on M^4 and X is a function on M^4 .

The Bianchi identities of the (massive) IIA supergravity give

$$\nabla^i W_i = 0 , \quad dS = S d\Phi , \quad dF = d\Phi \wedge F + S \star W , \quad (3.3)$$

where \star denotes the Hodge dual on M^4 . Furthermore, the field equations of (massive) IIA supergravity give

$$\begin{aligned} \nabla^2 \Phi &= -6A^{-1} \partial^i A \partial_i \Phi + 2(d\Phi)^2 + \frac{5}{4} S^2 + \frac{3}{8} F^2 - \frac{1}{2} W^2 + \frac{1}{4} X^2 , \\ dW &= -6A^{-1} dA \wedge W + 2d\Phi \wedge W + S \star F + XF , \\ \nabla^j F_{ij} &= -6A^{-1} \partial^j A F_{ij} + \partial^j \Phi F_{ij} + XW_i , \\ dX &= X(-6A^{-1} dA + d\Phi) , \end{aligned} \quad (3.4)$$

and the Einstein equation separates into an AdS component

$$\nabla^2 \ln A = -5\ell^{-2} A^{-2} - 6A^{-2} (dA)^2 + 2A^{-1} \partial_i A \partial^i \Phi + \frac{1}{4} X^2 + \frac{1}{4} S^2 + \frac{1}{8} F^2 , \quad (3.5)$$

which is interpreted as the field equation for the warp factor. The Bianchi identity $dS = Sd\Phi$ implies that if S is smooth, then either S is nowhere vanishing on M^4 or $S \equiv 0$ everywhere on M^4 . In what follows, we shall use the conventions and methodology of [17] for the investigation of AdS spaces where more details can be found.

3.1 The Killing Spinors

The solution of the KSEs of (massive) IIA supergravity along AdS_6 can be expressed as in (2.4), where σ_\pm and τ_\pm depend only on the coordinates of M^4 and are Majorana $Spin(9, 1)$

spinors that satisfy the lightcone projections $\Gamma_{\pm}\sigma_{\pm} = \Gamma_{\pm}\tau_{\pm} = 0$. The remaining KSEs have been stated in [17]. For the analysis which follows, it suffices to consider those acting on $\chi = \sigma_+$ and $\chi = \tau_+$. The IIA gravitino KSE implies that

$$\nabla_i \chi = \left(-\frac{1}{2}A^{-1}\partial_i A + \frac{1}{4}\Gamma_{(4)}\Gamma\mathcal{W}_i - \frac{1}{8}S\Gamma_i - \frac{1}{16}\not\Gamma_i\Gamma_{11} - \frac{1}{8}X\tilde{\Gamma}_{(4)}\Gamma_i \right) \chi, \quad (3.6)$$

where

$$\Gamma_{(4)} = \Gamma_{zyw}, \quad \Gamma_{ijkl} = \epsilon_{ijkl}\tilde{\Gamma}_{(4)}. \quad (3.7)$$

Furthermore the IIA dilatino KSE implies that

$$\left(\not\partial\Phi - \frac{1}{2}\mathcal{W}\Gamma_{(4)} + \frac{5}{4}S + \frac{3}{8}\not\Gamma_{11} + \frac{1}{4}X\tilde{\Gamma}_{(4)} \right) \chi = 0 \quad (3.8)$$

and there is a further algebraic KSE which arises during the integration of the KSEs along AdS_6 given by

$$\left(-\frac{c}{2\ell}A^{-1}\Gamma_z - \frac{1}{2}A^{-1}\not\partial A - \frac{1}{8}S - \frac{1}{16}\not\Gamma_{11} - \frac{1}{8}X\tilde{\Gamma}_{(4)} \right) \chi = 0, \quad (3.9)$$

where $c = 1$ for $\chi = \sigma_+$ and $c = -1$ for $\chi = \tau_+$. This is the analogue of the (2.8) in $D = 11$ supergravity. Note that (3.9) implies that σ_+ and τ_+ are linearly independent.

The counting of supersymmetries of IIA AdS_6 backgrounds proceeds as in the $D = 11$ case and so IIA AdS_6 backgrounds preserve either 16 or 32 supersymmetries. As there are not maximally supersymmetric AdS_6 backgrounds in IIA supergravity [18], it remains to investigate the AdS_6 backgrounds preserving 16 supersymmetries.

3.2 Proof of the main theorem

3.2.1 Orthogonality of σ_+ and τ_+ spinors

As in $D = 11$ supergravity, we proceed by setting

$$\Lambda = \sigma_+ + \tau_+, \quad (3.10)$$

then (3.6), together with the algebraic conditions (3.9) imply that

$$\nabla_i \parallel \Lambda \parallel^2 = \frac{2}{\ell}A^{-1}\langle \tau_+, \Gamma_i\Gamma_z\sigma_+ \rangle. \quad (3.11)$$

Furthermore, (3.6) also implies that

$$\nabla^i \langle \tau_+, \Gamma_i\Gamma_z\sigma_+ \rangle = -\langle \tau_+, \Gamma_i\Gamma_zA^{-1}\nabla^i A\sigma_+ \rangle. \quad (3.12)$$

On taking the divergence of (3.11) and using (3.12), we find

$$\nabla^2 \parallel \Lambda \parallel^2 + 2A^{-1}\nabla^i A\nabla_i \parallel \Lambda \parallel^2 = 0. \quad (3.13)$$

An application of the maximum principle implies that

$$\|\Lambda\|^2 = \text{const} . \quad (3.14)$$

For this, it is sufficient to require that the solutions are smooth and the internal space M^4 is closed.

Hence (3.11) implies that

$$\langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle = 0 . \quad (3.15)$$

Returning to the algebraic conditions (3.9); on taking inner products and making use of (3.15), we obtain

$$\begin{aligned} -\frac{1}{2\ell} A^{-1} \langle \tau_+, \sigma_+ \rangle - \frac{1}{8} S \langle \tau_+, \Gamma_z \sigma_+ \rangle - \frac{1}{16} \langle \tau_+, \Gamma_z \not{F} \Gamma_{11} \sigma_+ \rangle - \frac{1}{8} X \langle \tau_+, \Gamma_z \tilde{\Gamma}_{(4)} \sigma_+ \rangle &= 0 \\ \frac{1}{2\ell} A^{-1} \langle \sigma_+, \tau_+ \rangle - \frac{1}{8} S \langle \sigma_+, \Gamma_z \tau_+ \rangle - \frac{1}{16} \langle \sigma_+, \Gamma_z \not{F} \Gamma_{11} \tau_+ \rangle - \frac{1}{8} X \langle \sigma_+, \Gamma_z \tilde{\Gamma}_{(4)} \tau_+ \rangle &= 0 \end{aligned} \quad (3.16)$$

On subtracting these expressions, one obtains

$$\langle \sigma_+, \tau_+ \rangle = 0 . \quad (3.17)$$

This establishes the orthogonality of σ_+ and τ_+ spinors.

3.2.2 Additional properties of KSEs

To establish some additional properties of the KSEs observe that if σ_+ is a Killing spinor, then $\Gamma_{(4)} \sigma_+$ solves the KSEs as a τ_+ Killing spinor. This follows from the relation between σ_+ and τ_+ Killing spinors as explained in the context of 11-dimensional supergravity theory, section 2.2, that also applies in IIA supergravity. Then it follows from the orthogonality condition (3.17) of σ_+ and τ_+ spinors that

$$\langle \sigma_+, \Gamma_{(4)} \sigma_+ \rangle = 0 . \quad (3.18)$$

To proceed further, on eliminating the F terms between the algebraic KSEs (3.8) and (3.9), and using (3.18), one obtains the condition

$$\left(\frac{1}{6} \partial_i \Phi - \frac{1}{2} A^{-1} \partial_i A \right) \|\sigma_+\|^2 + \frac{1}{12} S \langle \sigma_+, \Gamma_i \sigma_+ \rangle = 0 . \quad (3.19)$$

The KSEs also imply that

$$\nabla_i \langle \sigma_+, \Gamma_{(4)} \sigma_+ \rangle = \left(\frac{1}{3} \partial_i \Phi - A^{-1} \partial_i A \right) \langle \sigma_+, \Gamma_{(4)} \sigma_+ \rangle - \frac{1}{6} W_i \|\sigma_+\|^2 + \frac{1}{6} S \langle \sigma_+, \Gamma_{(4)} \Gamma_i \sigma_+ \rangle , \quad (3.20)$$

which together with (3.18) gives

$$W_i \|\sigma_+\|^2 = S \langle \sigma_+, \Gamma_{(4)} \Gamma_i \sigma_+ \rangle . \quad (3.21)$$

There are then two cases to consider depending on whether $S \neq 0$ or $S \equiv 0$.

3.2.3 A non-existence theorem for standard IIA

In the special case for which S vanishes, $S \equiv 0$, (3.19) gives that

$$\frac{1}{6}d\Phi - \frac{1}{2}A^{-1}dA = 0 \quad (3.22)$$

and (3.21) implies that

$$W = 0 . \quad (3.23)$$

The dilaton field equation (3.4) then becomes

$$\nabla^2\Phi = \frac{3}{8}F^2 + \frac{1}{4}X^2 . \quad (3.24)$$

On integrating this expression over M^4 , one finds

$$F = 0, \quad X = 0 \quad (3.25)$$

and also

$$d\Phi = 0, \quad dA = 0 , \quad (3.26)$$

where again we have used that M^4 is closed. However, the warp factor equation (3.5) then admits no solution which establishes the non-existence theorem.

3.2.4 A non-existence theorem for massive IIA

As we have already mentioned a consequence of the Bianchi identity $dS = Sd\Phi$ is that S is nowhere vanishing. Furthermore, left-multiplying (3.8) with $\tilde{\Gamma}_{(4)}$, taking $\chi = \sigma_+$, and then taking the inner product with σ_+ , implies that

$$S\langle\sigma_+, \tilde{\Gamma}_{(4)}\sigma_+\rangle = -\frac{1}{5}X \parallel \sigma_+ \parallel^2 . \quad (3.27)$$

Also, taking the inner product of (3.8), with $\chi = \sigma_+$, and using (3.19), (3.21) and (3.27) to eliminate the spinor bilinear terms, one finds

$$-2(d\Phi)^2 + 6A^{-1}\partial_i A \partial^i \Phi - \frac{1}{2}W^2 + \frac{5}{4}S^2 - \frac{1}{20}X^2 = 0 . \quad (3.28)$$

Then, using (3.28) to eliminate the S^2 term, the dilaton field equation (3.4) can be rewritten as

$$\nabla^i \left(A^{12} \nabla_i (e^{-4\Phi}) \right) = -4A^{12}e^{-4\Phi} \left(\frac{3}{8}F^2 + \frac{3}{10}X^2 \right) . \quad (3.29)$$

Integrating this expression over M^4 and using the assumption that M^4 is closed yields the conditions

$$F = 0, \quad X = 0 , \quad (3.30)$$

and (3.29) then simplifies to

$$\nabla^2(e^{-4\Phi}) + 12A^{-1}\partial^i A\partial_i(e^{-4\Phi}) = 0 . \quad (3.31)$$

The maximum principle then implies that

$$d\Phi = 0 . \quad (3.32)$$

The algebraic KSE (3.9) then gives that

$$\frac{1}{8}S\sigma_+ = \left(-\frac{1}{2\ell}A^{-1}\Gamma_z - \frac{1}{2}A^{-1}\not\partial A \right)\sigma_+ , \quad (3.33)$$

and on squaring this expression we find

$$S^2 = 16\ell^{-2}A^{-2} + 16A^{-2}(dA)^2 . \quad (3.34)$$

On substituting (3.30), (3.32) and (3.34) into the Einstein equation (3.5), this condition can be rewritten as

$$\nabla^2 A^2 = -2\ell^{-2} . \quad (3.35)$$

However, this equation admits no regular solution. Hence there are no smooth closed supersymmetric warped AdS_6 solutions in massive IIA supergravity.

The assumptions mentioned in the description of the non-existence theorem are essential. First observe that the maximum principle has been used to establish that the length of the Killing spinor Λ is constant as in the 11-dimensional case. In addition, further partial integration arguments are required to establish the result, which require topological restrictions on the internal space. These assumptions can possibly be weakened, but not entirely removed. This is because one can reduce the $AdS_7 \times S^4$ solution of 11-dimensional supergravity, which has been interpreted as an AdS_6 solution in section (2.3.2), along Killing directions of S^4 to find AdS_6 solutions in IIA supergravity. However the existence of such solutions will not be a contradiction, as they do not satisfy the conditions of our theorem.

4 $AdS_6 \times_w M^4$ solutions in IIB supergravity

The non-vanishing form fluxes of IIB AdS_6 backgrounds have support on the internal space M^4 . In particular as the 5-form R-R field strength F is self-dual, it vanishes. The rest of the fields can be written as

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + A^2(dz^2 + e^{\frac{2z}{\ell}} \sum_{a=1}^3 (dx^a)^2) + ds^2(M^4) , \quad G = H , \quad P = \xi , \quad (4.1)$$

where G is a twisted⁶ complex 3-form which includes the R-R and the 3-form NS-NS 3-form field strengths, and P is the twisted 1-form field strength of the dilaton and axion

⁶In this formulation, the G is twisted with respect to the pull-back of the $U(1)$ bundle associated with the upper-half plane $SU(1,1)/U(1)$ which is the target space of the IIB sigma model scalars.

of the theory. Thus H and ξ are twisted complex 3- and 1-forms on the internal space M^4 , respectively. We also introduce the frame $(\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^z, \mathbf{e}^a, \mathbf{e}^i)$ as in (2.6) and ℓ is the radius of AdS_6 . We follow the conventions, notation and methodology of [16] for investigating IIB AdS backgrounds.

To continue it is convenient to define

$$W_i = \frac{1}{6} \epsilon_i^{jkl} H_{jkl} , \quad (4.2)$$

where W is a complex (twisted) 1-form on M^4 . In such a case the Bianchi identities and the field equations that we shall use below can be expressed as

$$\nabla^i W_i = iQ_i W^i - \xi_i \bar{W}^i , \quad d\xi = 2iQ \wedge \xi , \quad dQ = -i\xi \wedge \bar{\xi} , \quad (4.3)$$

and

$$\begin{aligned} \nabla_{[i} W_{j]} &= -6\partial_{[i} \log A W_{j]} + iQ_{[i} W_{j]} + \xi_{[i} \bar{W}_{j]} , \\ \nabla^i \xi_i &= -6\partial^i \log A \xi_i + 2iQ^i \xi_i - \frac{1}{4} W^2 , \\ A^{-1} \nabla^2 A &= \frac{1}{8} \|W\|^2 - \frac{5}{\ell^2} A^{-2} - 5(d \log A)^2 , \\ R_{ij}^{(4)} &= 6A^{-1} \nabla_i \nabla_j A + \frac{3}{8} |W|^2 \delta_{ij} - \frac{1}{2} W_{(i} \bar{W}_{j)} + 2\xi_{(i} \bar{\xi}_{j)} , \end{aligned} \quad (4.4)$$

respectively, where Q is the connection of the $U(1)$ bundle on the upper-half plane pulled back on M^4 .

4.1 The Killing spinors

The KSEs of IIB supergravity can be solved along AdS_6 [16] to yield an expression for the Killing spinor as in (2.4), where now σ_\pm and τ_\pm are complex Weyl $Spin(9, 1)$ spinors that depend only on the coordinates of M^4 and satisfy the projections $\Gamma_\pm \sigma_\pm = \Gamma_\pm \tau_\pm = 0$.

Next as in the IIA case define

$$\Gamma_{(4)} = \Gamma_{zyxw} , \quad \Gamma_{ijkl} = \epsilon_{ijkl} \tilde{\Gamma}_{(4)} , \quad (4.5)$$

with the convention that

$$\Gamma_{(4)} \tilde{\Gamma}_{(4)} \sigma_\pm = \pm \sigma_\pm , \quad (4.6)$$

and similarly for τ_\pm , as these spinors. Using these conventions, the remaining independent KSEs are

$$\nabla_i^{(\pm)} \sigma_\pm = 0 , \quad \nabla_i^{(\pm)} \tau_\pm = 0 , \quad \mathcal{A}^{(\pm)} \sigma_\pm = 0 , \quad \mathcal{A}^{(\pm)} \tau_\pm = 0 , \quad (4.7)$$

and

$$\Xi_\pm \sigma_\pm = 0 , \quad \left(\Xi_\pm \pm \frac{1}{\ell} \right) \tau_\pm = 0 , \quad (4.8)$$

where

$$\nabla_i^{(\pm)} = \nabla_i + \Psi_i^{(\pm)} , \quad (4.9)$$

with

$$\begin{aligned} \Psi_i^{(\pm)} &= \pm \frac{1}{2} \partial_i \log A - \frac{i}{2} Q_i - \left(\frac{1}{16} W_i \tilde{\Gamma}_{(4)} + \frac{3}{16} \Gamma^j_i W_j \tilde{\Gamma}_{(4)} \right) C * , \\ \Xi_{\pm} &= \mp \frac{1}{2\ell} - \frac{1}{2} \Gamma_z \partial_i A \Gamma^i + \frac{1}{16} A \Gamma_z \mathcal{W} \tilde{\Gamma}_{(4)} C * , \quad \mathcal{A}^{(\pm)} = \frac{1}{4} \mathcal{W} \tilde{\Gamma}_{(4)} + \mathcal{Z} C * , \end{aligned} \quad (4.10)$$

and ∇ is the Levi-Civita connection on M^4 . The (4.7) KSEs are a suitable restriction of the gravitino and dilatino KSEs of IIB supergravity on τ_{\pm} and σ_{\pm} while the (4.8) KSEs arise as integrability conditions of the solution of IIB KSEs along AdS_6 . Observe also that these KSEs imply that the σ_+ and τ_+ Killing spinors are linearly independent.

The counting of Killing spinors is similar as that presented in more detail for AdS_6 solutions of 11-dimensional supergravity and so IIB AdS_6 backgrounds preserve either 16 or 32 supersymmetries. It is known for sometime that there are no IIB AdS_6 backgrounds [18]. Therefore it remains to explore the IIB AdS_6 backgrounds that preserve 16 supersymmetries.

4.2 Non-existence theorems in IIB

4.2.1 The orthogonality of τ_+ and σ_+ Killing spinors

As in previous cases setting $\Lambda = \sigma_+ + \tau_+$ and upon using the gravitino KSE, one finds

$$\nabla_i \parallel \Lambda \parallel^2 = -\partial_i \log A \parallel \Lambda \parallel^2 + \frac{1}{8} \text{Re} \langle \Lambda, W_i \tilde{\Gamma}_{(4)} C * \Lambda \rangle . \quad (4.11)$$

Next the algebraic KSE (4.8) gives

$$-\partial_i \log A \parallel \Lambda \parallel^2 + \frac{1}{8} \text{Re} \langle \Lambda, W_i \tilde{\Gamma}_{(4)} C * \Lambda \rangle = 2\ell^{-1} A^{-1} \text{Re} \langle \tau_+, \Gamma_{iz} \sigma_+ \rangle . \quad (4.12)$$

Thus, one finds

$$\nabla_i \parallel \Lambda \parallel^2 = 2\ell^{-1} A^{-1} \text{Re} \langle \tau_+, \Gamma_{iz} \sigma_+ \rangle . \quad (4.13)$$

Furthermore observe that as a consequence of the gravitino KSE in (4.7), one has

$$\nabla^i (A \text{Re} \langle \tau_+, \Gamma_{iz} \sigma_+ \rangle) = 0 . \quad (4.14)$$

Taking the divergence of (4.13) and using the above equation, one deduces that

$$\nabla^2 \parallel \Lambda \parallel^2 + 2A^{-1} \partial^i A \partial_i \parallel \Lambda \parallel^2 = 0 . \quad (4.15)$$

The maximum principle then gives that $\parallel \Lambda \parallel$ is constant and in turn

$$\begin{aligned} -\partial_i \log A \parallel \Lambda \parallel^2 + \frac{1}{8} \text{Re} \langle \Lambda, W_i \tilde{\Gamma}_{(4)} C * \Lambda \rangle &= 0 , \\ \text{Re} \langle \tau_+, \Gamma_{iz} \sigma_+ \rangle &= 0 , \end{aligned} \quad (4.16)$$

where it is sufficient to assume the smoothness of the fields and Killing spinors, and that M^4 is closed. Furthermore the algebraic KSEs (4.8) give that

$$\text{Re} \langle \tau_+, \sigma_+ \rangle = 0 . \quad (4.17)$$

This establishes the orthogonality of τ_+ and σ_+ spinors.

4.2.2 A formula for the warp factor

Observe that as a consequence of the first equation in (4.16) as well as the orthogonality condition in (4.17) that

$$\text{Re} \langle \sigma'_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle = 0, \quad \text{Re} \langle \tau_+, W_i \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle = 0, \quad (4.18)$$

where we have used that $\tilde{\Gamma}_{(4)} \sigma_+$ is a τ_+ Killing spinor because of (4.6) and the relation between τ_+ and σ_+ spinors.

Taking the derivative of the first equations in (4.16), we obtain for $\Lambda = \sigma_+$ that

$$\begin{aligned} & -\nabla^2 \log A \parallel \sigma_+ \parallel^2 - (\partial_i \log A)^2 \parallel \sigma_+ \parallel^2 + \frac{1}{64} |W|^2 \parallel \sigma_+ \parallel^2 \\ & + \frac{3}{64} \text{Re} \langle \sigma_+, \bar{W}^i W^j \Gamma_{ij} \sigma_+ \rangle - \frac{1}{8} \text{Re} \langle \sigma_+, \xi_i \bar{W}^i \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle = 0. \end{aligned} \quad (4.19)$$

Moreover the dilatino KSE in (4.7) can be used to give

$$\frac{1}{4} |W|^2 \parallel \sigma_+ \parallel^2 + \frac{1}{4} \text{Re} \langle \sigma_+, \bar{W}^i W^j \Gamma_{ij} \sigma_+ \rangle + \text{Re} \langle \sigma_+, \xi_i \bar{W}^i \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle = 0. \quad (4.20)$$

Eliminating the term that contains ξ in the above two equations, we find

$$\begin{aligned} & -\nabla^2 \log A \parallel \sigma_+ \parallel^2 - (\partial_i \log A)^2 \parallel \sigma_+ \parallel^2 \\ & + \frac{3}{64} |W|^2 \parallel \sigma_+ \parallel^2 + \frac{5}{64} \text{Re} \langle \sigma_+, \bar{W}^i W^j \Gamma_{ij} \sigma_+ \rangle = 0. \end{aligned} \quad (4.21)$$

This is the key equation that we shall explore in what follows to establish non-existence theorems. Observe that apart from the last term, it has the required form to apply the maximum principle on the warp factor A .

4.2.3 A non-existence theorem for linearly dependent NSNS and RR fluxes

IIB backgrounds have linearly dependent NSNS and RR fluxes if there exist nowhere vanishing (twisted) complex function f on M^4 such that

$$\text{Re}(f W_i) = 0. \quad (4.22)$$

This is the case for all IIB backgrounds that have either NSNS or RR 3-form fluxes and all their $SL(2, \mathbb{Z})$ duals. For backgrounds for which the fluxes satisfy (4.22), the last term in (4.21) vanishes. This follows easily after solving (4.22) as

$$W_i = -\frac{\bar{f}^2}{|f|^2} \bar{W}_i \quad (4.23)$$

and substituting into

$$\text{Re} \langle \sigma_+, \bar{W}^i W^j \Gamma_{ij} \sigma_+ \rangle = \text{Re} \langle \sigma_+, \left(-\frac{\bar{f}^2}{|f|^2} \right) \bar{W}^i \bar{W}^j \Gamma_{ij} \sigma_+ \rangle = 0. \quad (4.24)$$

As the last term in (4.21) vanishes an application of the maximum principle reveals that the warp factor is constant and $W = 0$. This in turn makes the field equation for the warp factor A (4.4) inconsistent and so such backgrounds do not exist.

Furthermore, it is required that smooth closed AdS_6 solutions must have active scalars. Indeed if $\xi = 0$, then the algebraic KSE in (4.7) implies that

$$|W|^2 \parallel \sigma_+ \parallel^2 + \text{Re} \langle \sigma_+, \bar{W}^i W^j \Gamma_{ij} \sigma_+ \rangle = 0 , \quad (4.25)$$

which can be used to eliminate the last term in (4.21). Then the maximum principle can apply to yield that A is constant and $W = 0$ which in turn are inconsistent with the field equation for the warp factor A (4.4).

4.2.4 A non-existence theorem for $\langle \sigma_+, C * \sigma_+ \rangle \neq 0$

To demonstrate this non-existence theorem for smooth closed IIB AdS_6 solutions, we shall first show that either the bilinear $\langle \sigma_+, C * \sigma_+ \rangle$ vanishes identically or it does not vanish anywhere on the internal space. To begin, we evaluate the derivative of this bilinear and after using the gravitino in (4.7) and algebraic (4.8) KSEs, to find that

$$\nabla_i (A^4 \langle \sigma_+, C * \sigma_+ \rangle) = -i Q_i A^4 \langle \sigma_+, C * \sigma_+ \rangle . \quad (4.26)$$

This is a parallel transport equation which establishes the statement as A does not vanish anywhere on the internal manifold.

Next observe that the second equation in (4.18) can be written as

$$\text{Re} \langle \sigma_+, W_i C * \sigma_+ \rangle = 0 . \quad (4.27)$$

This condition is as that in (4.22) with $f = \langle \sigma_+, C * \sigma_+ \rangle$. Therefore if $\langle \sigma_+, C * \sigma_+ \rangle$ does not vanish identically, then the non-existence theorem follows from the results of the previous section. Observe that $\langle \sigma_+, C * \sigma_+ \rangle$ satisfies the requirements on the function f .

Incidentally observe that if the bi-linear $\alpha = \langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle$ does not vanish anywhere on M^4 , then there are no smooth closed solutions. Indeed from (4.16), one finds that

$$W_i = \frac{16}{|\alpha|^2} \partial_i \log A \parallel \sigma_+ \parallel^2 - \frac{\bar{\alpha}^2}{|\alpha|^2} \bar{W}_i . \quad (4.28)$$

Substituting this into (4.21), the latter can be rearranged such that the maximum principle applies leading to $W = 0$ which together with the field equations for the warp factor A establishes the statement.

4.2.5 A non-existence theorem for $\det \omega \neq 0$

So far we have seen that smooth closed IIB AdS_6 backgrounds exist provided that

$$\langle \sigma_+, C * \sigma_+ \rangle = 0 , \quad (4.29)$$

and α vanishes somewhere on the internal space. However, there are additional restrictions on the spinor bilinears required for the existence of such AdS_6 backgrounds. To find them,

let us explore further the geometry of the internal space. A more systematic investigation of the spinor bilinears can be found in appendices B and D.

To continue define

$$\omega_{ij} = \langle \sigma_+, \Gamma_{ij} \sigma_+ \rangle, \quad {}^* \omega_{ij} = \langle \sigma_+, \Gamma_{ij} \tilde{\Gamma}_{(4)} \sigma_+ \rangle. \quad (4.30)$$

Observe that $\langle \sigma_+, \Gamma_{ij} \tilde{\Gamma}_{(4)} \sigma_+ \rangle = -\frac{1}{2} \epsilon_{ij}{}^{kl} \langle \sigma_+, \Gamma_{kl} \sigma_+ \rangle$. An application of the gravitino KSE reveals that

$$\begin{aligned} \nabla_k \omega_{ij} &= -\partial_k \log A \langle \sigma_+, \Gamma_{ij} \sigma_+ \rangle - \frac{3i}{8} [\text{Im} \langle \sigma_+, W_i \delta_{kj} \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle - (j, i)] , \\ \nabla_k {}^* \omega_{ij} &= -\partial_k \log A \langle \sigma_+, \Gamma_{ij} \tilde{\Gamma}_{(4)} \sigma_+ \rangle - \frac{3i}{8} \epsilon_{ijk}{}^m \text{Im} \langle \sigma_+, W_m \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle . \end{aligned} \quad (4.31)$$

These in particular imply that

$$d(A\omega) = 0, \quad \nabla_i (A^* \omega_{jk}) = \nabla_{[i} (A^* \omega_{jk]) , \quad (4.32)$$

ie $A^* \omega$ is a Killing-Yano tensor.

Furthermore, it is easy to show using the equation (4.31) that

$$\nabla_k (A^2 \omega_{ij} {}^* \omega^{ij}) = 0. \quad (4.33)$$

Thus $A^2 \omega_{ij} {}^* \omega^{ij} = \text{const}$. If this constant does not vanish, then ω has rank 4 everywhere on the internal manifold M^4 .

On the other hand using (4.29), one can show from the dilatino KSE that

$$W^i {}^* \omega_{ij} = 0. \quad (4.34)$$

Thus if $A^2 \omega_{ij} {}^* \omega^{ij} \neq 0$, and so both ${}^* \omega$ and ω are invertible, then W will vanish everywhere on M^4 . Again in such a case there are no smooth closed AdS_6 solutions. This establishes our theorem. Therefore if smooth closed AdS_6 solutions exist, they require that ω has rank 2 or less everywhere on the internal space.

4.3 Symmetries and topology of internal manifold

Further restrictions on the internal space of smooth closed IIB AdS_6 backgrounds can be found by exploring the isometries of AdS_6 backgrounds. We have established in appendix A that the internal space of AdS_6 backgrounds admits an $\mathfrak{so}(3)$ action generated by three⁷ vector fields

$$Z_i^Q = A \text{Re} \langle \sigma_+, Q \Gamma_i \sigma_+ \rangle, \quad (4.35)$$

for $Q = \Gamma_{z23}, -\Gamma_{z13}, \Gamma_{z12}$. It is then a consequence of the KSEs that this action leaves all the fields invariant.

⁷In fact there is the possibility of a fourth for $Q = \Gamma_{123}$ but this vanishes identically because of (4.16).

First let us demonstrate that this action is non-trivial. For this we shall show that the Z_Q cannot vanish identically for smooth closed solutions. To see this first note that the Fierz identities in appendix B give

$$\begin{aligned}\omega^2 - 2|\alpha|^2 + 2(\|\sigma_+\|^2)^2 &= 0, \\ \sum_Q Z_Q^2 + 2A^2|\alpha|^2 - 2A^2(\|\sigma_+\|^2)^2 &= 0,\end{aligned}\tag{4.36}$$

where $\alpha = \langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle$ as in previous sections and we have used that $\langle \sigma_+, C * \sigma_+ \rangle = \langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle = 0$.

Now suppose that there are smooth closed solutions for which Z_r vanish identically on the internal space. Then as a consequence of the Fierz identities $\omega^2 = 0$ and so $\omega = 0$. On the other hand if ω vanishes, then an application of the maximum principle on (4.21) implies that there are no smooth closed AdS_6 solutions which is a contradiction. This establishes that for smooth closed solutions $\mathfrak{so}(3)$ acts non-trivially on the internal space.

The groups with Lie algebra $\mathfrak{so}(3)$ act on 4-dimensional manifolds and the principal⁸ orbits are either of co-dimension 1 or 2. To continue, let us suppose that the principal orbits of $\mathfrak{so}(3)$ on M^4 have co-dimension 1. As all fluxes are invariant up to a possible gauge $U(1)$ transformation and $Z_Q^i W_i = 0$ (A.2), $W \wedge \bar{W}$ descends on the space of orbits. As the space of orbits is 1-dimensional $W \wedge \bar{W}$ vanishes. Then we can apply the maximum principle on (4.21) to conclude that the warp factor is constant and $W = 0$ which lead to an inconsistency in the field equation for the warp factor. Thus, *there are no smooth closed AdS_6 backgrounds with principal $\mathfrak{so}(3)$ orbits on the internal space of co-dimension 1.*

It remains to investigate the case that for which the principal orbits of $\mathfrak{so}(3)$ action on M^4 have co-dimension 2. For this let us assume that the $\mathfrak{so}(3)$ action on M^4 can be integrated to an effective $SU(2)$ or $SO(3)$ on M^4 , and that M^4 is oriented. Oriented 4-dimensional manifolds admitting an effective non-abelian group action have been classified in [25] and references within, see also [27]. The principal orbits are 2-spheres. These can degenerate to either points or to the 2-dimensional real projective space $\mathbb{R}P^2$. The space of orbits is a 2-dimensional surface F with boundary which is a union of circles. The principal orbits degenerate at the boundary circles. Spin oriented 4-dimensional manifolds M^4 which admit an effective $SU(2)$ or $SO(3)$ with principal co-dimension 2 orbits include⁹ those presented in (1.1). For $M(F)$, the surface F has no boundary and the principal orbits do not degenerate. If F is oriented, then $M(F) = F \times S^2$. However we shall demonstrate later that there are no smooth closed AdS_6 backgrounds with internal space M^4 that has topology $F \times S^2$.

4.4 Aspects of geometry

To explore the geometry of the internal space of IIB AdS_6 backgrounds, let us assume that $\langle \sigma_+, C * \sigma_+ \rangle = \langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle = 0$ and $\det \omega = 0$. Observe that (4.34) can be rewritten

⁸These are the orbits with the smallest isotropy group.

⁹There are some other possibilities, see appendix C.

as

$$W \wedge \omega = 0 , \quad (4.37)$$

and the algebraic and dilatino KSEs give

$$\xi \wedge \omega = dA \wedge \omega = 0 . \quad (4.38)$$

The 2-form $A\omega$ descends on the space of orbits F as $\mathcal{L}_{Z^Q} A\omega = 0$ and $i_{Z^Q} \omega = 0$. The latter follows from (B.3).

Note that ω gives rise to a Hermitian structure on F away from the fixed points. This can be seen from the Fierz identity (B.5) which can be re-written as

$$A^2 \omega_{ik} \omega_j^k = \sum_Q Z_i^Q Z_j^Q + A^2 \left(|\alpha|^2 - (\| \sigma_+ \|^2)^2 \right) \delta_{ij} . \quad (4.39)$$

If there are points for which $|\alpha| = \| \sigma_+ \|^2$, ω is not a Hermitian form on F . Incidentally, the points for which $|\alpha| = \| \sigma_+ \|^2$ are the fixed points of the group action as can be seen from (B.4).

In addition, setting as before $\alpha = \langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle$, one finds that

$$\nabla_i \alpha = -\partial_i \log A \alpha - i Q_i \alpha + \frac{1}{8} \bar{W}_i \| \sigma_+ \|^2 + \frac{3}{8} \bar{W}_m \omega^m_i . \quad (4.40)$$

Using the algebraic KSE Ξ to eliminate the last term, we find that

$$\nabla_i (A^4 \alpha) = -i Q_i A^4 \alpha + \frac{1}{2} A^4 \bar{W}_i \| \sigma_+ \|^2 - \frac{3}{\ell} A^3 \langle \sigma_+, \Gamma_z \Gamma_i \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle . \quad (4.41)$$

This condition can be solved for W to determine the flux in terms of the geometry of the internal space.

4.5 A non-existence theorem for $M^4 = F \times S^2$

We shall now demonstrate that there are no smooth closed AdS_6 solutions for which the internal space has topology¹⁰ $M^4 = F \times S^2$, where F is an oriented surface. To see this, observe that in this case the action of $\mathfrak{so}(3)$ has no fixed points. The orbits are 2-spheres and all are principal. As a result, there are no points in M^4 for which all Killing vector fields Z^Q vanish simultaneously. Then the second equation in (4.36) implies that

$$|\alpha| - \| \sigma_+ \|^2 < 0 . \quad (4.42)$$

In turn the first equation in (4.36) implies that ω is nowhere vanishing. As A is nowhere vanishing, the 2-form $A^n \omega$ is nowhere vanishing as well for $n \in \mathbb{Z}$. Furthermore $A^n \omega$ descends on F and as $A^n \omega$ is nowhere vanishing, one has

$$\int_F A^n \omega \neq 0 . \quad (4.43)$$

¹⁰It is allowed though for the metric on M^4 to be a warped product of the metric of S^2 with that of F .

On the other hand, an application of the gravitino KSE (4.7) reveals that

$$\begin{aligned} \nabla_i \langle \sigma_+, \Gamma_{aj} \sigma_+ \rangle &= -\partial_i \log A \langle \sigma_+, \Gamma_{aj} \sigma_+ \rangle + \frac{i}{8} \text{Im} \langle \sigma_+, W_i \Gamma_{aj} \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle \\ &+ \frac{3i}{8} \text{Im} \langle \sigma_+, W_j \Gamma_{ai} \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle - \frac{3i}{8} \delta_{ij} \text{Im} \langle \sigma_+, \Gamma_a W_k \Gamma^k \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle , \end{aligned} \quad (4.44)$$

for $a = 1, 2, 3, z$. Using this and the KSE (4.8) we obtain

$$A^2 \omega = \frac{\ell}{2} d(A^3 \zeta) , \quad (4.45)$$

where $\zeta_i = \langle \sigma_+, \Gamma_{zi} \sigma_+ \rangle$. Pulling this equation back on F with a global section of the trivial fibration $F \times S^2$, we find upon application of Stoke's theorem that

$$\int_F A^2 \omega = 0 . \quad (4.46)$$

This is in contradiction to (4.43) for $n = 2$. We therefore conclude that there are no smooth closed AdS_6 solutions with internal manifolds that have topology $F \times S^2$.

4.6 Conditions for the existence of AdS_6 IIB solutions

We have seen that the existence of smooth closed IIB AdS_6 backgrounds is rather restricted. However, we have not been able to establish as strong a result as that for $D = 11$ and (massive) IIA backgrounds. If smooth closed IIB AdS_6 backgrounds exist, the statements of non-existence theorems that we have established can turn to necessary conditions that such backgrounds must satisfy. Of course the issue of existence of smooth closed IIB AdS_6 backgrounds can be settled with an example. However to our knowledge all solutions that have been found in the literature [3]-[9] so far are singular. In particular those found in [9] are singular because the warp factor A of the AdS_6 subspace vanishes at some points on the internal space. As it has been shown in [16] that irrespective of the frame chosen, A is related to the length of a Killing spinor [16], $\| \sigma_- \|^2 = c^2 A^2$, where c is a constant. For smooth closed AdS_6 backgrounds, the Killing spinors must be nowhere vanishing, and so $A \neq 0$ everywhere on the internal space. Here we shall summarize the necessary conditions for existence of smooth closed IIB AdS_6 backgrounds.

We have seen that the bilinears of Killing spinor σ_+ must satisfy the following conditions

$$\langle \sigma_+, C * \sigma_+ \rangle = 0 , \quad \det \omega = 0 , \quad \langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle = 0 . \quad (4.47)$$

Violation of either of the first two conditions leads to a non-existence result while the third condition is a consequence of the orthogonality of σ_+ and τ_+ Killing spinors.

Furthermore the bilinear

$$\langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle , \quad (4.48)$$

must vanish at some points of the internal space M^4 . Otherwise again a non-existence theorem for smooth closed AdS_6 backgrounds can be established.

Another necessary condition for the existence of smooth closed IIB AdS_6 backgrounds is that they must have active scalars. If the IIB scalars are constant, then one can demonstrate that there is a non-existence theorem for such solutions.

In addition, it has been shown in appendix D that for smooth closed AdS_6 backgrounds, the following spinor bilinears also vanish

$$\langle \sigma_+, \Gamma_{12}\sigma_+ \rangle = \langle \sigma_+, \Gamma_{13}\sigma_+ \rangle = \langle \sigma_+, \Gamma_{23}\sigma_+ \rangle = 0 , \quad (4.49)$$

otherwise such backgrounds cannot exist. This is equivalent to the statement that all σ -type Killing spinors are orthogonal over the complex numbers. This is despite the fact that the KSEs of IIB supergravity are linear over the real numbers.

Moreover we have demonstrated that if smooth closed IIB AdS_6 solutions exist, the internal space has principal orbits of $\mathfrak{so}(3)$ of co-dimension 2. Otherwise again a non-existence theorem can be established. Combining this with the classification results of [25], it is possible to determine the diffeomorphic type of the internal space that includes manifolds as those in (1.1). Using these classification results, we also excluded the existence of smooth closed IIB AdS_6 solutions with internal space which has topology $F \times S^2$, where F is an oriented surface. This restricts further the possible topologies of smooth closed IIB AdS_6 backgrounds to S^4 , $p(S^1 \times S^3) \# q(S^1 \times \mathbb{R}P^3)$ and $M(F)$, where F is a non-oriented surface.

Amongst these, an interesting topology for an internal space is $p(S^1 \times S^3) \# q(S^1 \times \mathbb{R}P^3)$. The space of orbits is a surface F with boundary ∂F which is a union of circles. If n is the number of circles that S^2 degenerates to a point, then $q = \text{rank} H_0(\partial F) - n$ and $p = \text{rank} H_1(F) - q$. For $q = 0$ and $p = 1$, F is a cylinder as expected and $n = 2$, while for $q = 1$ and $p = 0$ again F is a cylinder but now $n = 1$. The internal spaces of examples explored in [8, 9] may have such topology.

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Appendix A Symmetries

A.1 Killing Vector fields

Let us define the 1-form spinor bilinears as in (4.35). Then using the gravitino KSE, one finds

$$\nabla_i Z_j^Q = -\frac{3}{8} A \epsilon_{ij}{}^{kl} \text{Re} \langle \Lambda, W_k Q \Gamma_l C * \Lambda \rangle , \quad (A.1)$$

where $Q = \Gamma_{23}, -\Gamma_{13}, \Gamma_{12}$. Potentially, there is an additional vector bilinear for $Q = \Gamma_{123}$ but it vanishes identically because of (4.16). However note that to derive (4.16) one uses the maximum principle and so the vanishing of this vector field requires a global condition.

It follows directly that the associated vector fields are Killing. Furthermore, one finds from the algebraic KSEs that

$$Z_k^Q W^k = 0, \quad Z_k^Q \bar{\xi}^k = 0, \quad Z_k^Q \partial^k A = 0. \quad (\text{A.2})$$

Next after using the Einstein equation, one can show that

$$\nabla^2 |Z^Q|^2 = \frac{1}{2} |dZ^Q|^2 - 6 \partial^k \log A \partial_k |Z^Q|^2 - \frac{3}{4} |W|^2 |Z^Q|^2. \quad (\text{A.3})$$

Notice however that the right-hand-side of this equation is not definite to apply the maximum principle.

The algebra of the three vector fields is

$$[Z^{Q_1}, Z^{Q_2}] = -\frac{3}{\ell} \|\sigma_+\|^2 Z^{Q_3}, \quad (\text{A.4})$$

where $Q_1 = \Gamma_{z23}, Q_2 = -\Gamma_{z13}, Q_3 = \Gamma_{z12}$ and cyclic in the Q 's for the other commutators. Therefore it is isomorphic to $\mathfrak{so}(3)$. To prove this one uses (A.1), (A.2), the algebraic KSE Ξ in (4.8) on σ_+ as well as (B.4). This is significant as 4-dimensional manifolds admitting effective non-abelian group actions have been classified.

A.2 Notation and conventions

Our form conventions are as follows. Let ω be a k -form, then

$$\omega = \frac{1}{k!} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad \omega_{ij}^2 = \omega_{i\ell_1 \dots \ell_{k-1}} \omega_j^{\ell_1 \dots \ell_{k-1}}, \quad \omega^2 = \omega_{i_1 \dots i_k} \omega^{i_1 \dots i_k}. \quad (\text{A.5})$$

We also define

$$\phi = \omega_{i_1 \dots i_k} \Gamma^{i_1 \dots i_k}, \quad \phi_{i_1} = \omega_{i_1 i_2 \dots i_k} \Gamma^{i_2 \dots i_k}, \quad \not{I}\omega_{i_1} = \Gamma_{i_1}^{i_2 \dots i_{k+1}} \omega_{i_2 \dots i_{k+1}}, \quad (\text{A.6})$$

where the Γ_i are the Dirac gamma matrices.

The inner product $\langle \cdot, \cdot \rangle$ we use on the space of spinors is that for which space-like gamma matrices are Hermitian while time-like gamma matrices are anti-hermitian, ie the Dirac spin-invariant inner product is $\langle \Gamma_0 \cdot, \cdot \rangle$. For more details on our conventions see [15, 17, 16].

B IIB Fierz Identities

In the investigation of the geometry of the internal spaces of IIB AdS_6 backgrounds, we have used some Fierz identities. These are listed below without assuming the vanishing of any of the spinor bilinears which are necessary for the existence of smooth closed IIB AdS_6 backgrounds. So the identities below are valid in general. However, we shall comment on their consequences in the special cases of smooth closed IIB AdS_6 backgrounds.

The first Fierz identity gives ω^2 in terms of scalar bilinears as

$$\begin{aligned} \langle \sigma_+, \Gamma_{ij} \sigma_+ \rangle \langle \sigma_+, \Gamma^{ij} \sigma_+ \rangle &= 2 |\langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle|^2 - 2 (\|\sigma_+\|^2)^2 \\ &- 2 |\langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle|^2 + 2 |\langle \sigma_+, C * \sigma_+ \rangle|^2. \end{aligned} \quad (\text{B.1})$$

Observe that the length of ω can vanish for smooth closed IIB AdS_6 backgrounds whenever $\tilde{\Gamma}_{(4)}C * \sigma_+ = \sigma_+$ as a consequence of the Cauchy-Schwarz inequality.

Next, one finds that

$$\begin{aligned} \epsilon^{ijmn} \langle \sigma_+, \Gamma_{ij} \sigma_+ \rangle \langle \sigma_+, \Gamma_{mn} \sigma_+ \rangle &= -8 \text{Re} \left(\langle C * \sigma_+, \sigma_+ \rangle \langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle \right) \\ &+ 8 \| \sigma_+ \|^2 \langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle . \end{aligned} \quad (\text{B.2})$$

For smooth closed IIB AdS_6 solutions, this implies that ω must have rank strictly less than 4. This was derived already using a different reasoning.

The Fierz identity

$$\begin{aligned} \langle \sigma_+, \Gamma_{ij} \sigma_+ \rangle \langle \sigma_+, Q \Gamma^j \sigma_+ \rangle &= \frac{1}{2} \langle C * \sigma_+, \sigma_+ \rangle \langle \sigma_+, Q \Gamma_i C * \sigma_+ \rangle \\ &- \frac{1}{2} \langle \sigma_+, C * \sigma_+ \rangle \langle Q \Gamma_i C * \sigma_+, \sigma_+ \rangle \\ &+ \langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle \langle \sigma_+, \tilde{\Gamma}_{(4)} Q \Gamma_i \sigma_+ \rangle , \end{aligned} \quad (\text{B.3})$$

is instrumental to show that $A\omega$ descends on the space of orbits of $\mathfrak{so}(3)$ action on the internal space M^4 for smooth closed backgrounds.

The Fierz identity

$$\begin{aligned} \sum_{a=1}^4 \langle \sigma_+, Q_a \Gamma_i \sigma_+ \rangle \langle \sigma_+, Q_a \Gamma^i \sigma_+ \rangle + 2 |\langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle|^2 - 2 (\| \sigma_+ \|^2)^2 \\ + 2 \langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle^2 - 2 |\langle \sigma_+, C * \sigma_+ \rangle|^2 = 0 , \end{aligned} \quad (\text{B.4})$$

has been instrumental in showing that the $\mathfrak{so}(3)$ acts non-trivially on the internal space of smooth closed AdS_6 backgrounds.

The Fierz identity

$$\begin{aligned} \langle \sigma_+, \Gamma_{ik} \sigma_+ \rangle \langle \sigma_+, \Gamma_j^k \sigma_+ \rangle &= \sum_{a=1}^4 \langle \sigma_+, Q_a \Gamma_i \sigma_+ \rangle \langle \sigma_+, Q_a \Gamma_j \sigma_+ \rangle \\ &+ \left(|\langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle|^2 - (\| \sigma_+ \|^2)^2 \right) \delta_{ij} \end{aligned} \quad (\text{B.5})$$

has been used to investigate the existence of a Hermitian structure on the space of orbits of the $\mathfrak{so}(3)$ action on the internal space M^4 .

C 4-Manifolds with Large Symmetry Groups

It has been known for sometime [25] that closed oriented 4-dimensional manifolds that support an effective action of $SO(3)$ or $SU(2)$ are diffeomorphic to

1. S^4 or $\pm \mathbb{C}P^2$,
2. connected sums $p(S^1 \times S^3) \# q(S^1 \times \mathbb{R}P^3)$,

3. bundles over S^1 with fibre $SU(2)/D$, where D is a discrete subgroup,
4. bundles over surfaces with fibre S^2 , and
5. certain quotients of S^2 -bundles over surfaces with involutions.

From these those collected in (1.1) are spin, closed, and $SO(3)$ or $SU(2)$ have co-dimension 2 principal orbits. The $SO(3)$ or $SU(2)$ group action on $M(F)$ have orbits 2-spheres S^2 and are all principal. $M(F)$ is uniquely characterized as the 2-sphere bundle over a surface F that admits a spin structure. $M(F) = F \times S^2$, if F is oriented. The spaces $p(S^1 \times S^3) \# q(S^1 \times \mathbb{R}P^3)$ arise in the case that the principal orbit S^2 degenerates to several other special orbits. For oriented 4-manifolds S^2 either collapses to a point or degenerates to $\mathbb{R}P^2$. In both cases, the special orbits are not isolated but rather lie on a circle. In particular in $S^1 \times S^3$ the S^2 fibre collapses to a point at two different circles, while in $S^1 \times \mathbb{R}P^3$ the S^2 fibre collapses to point at a circle and becomes $\mathbb{R}P^2$ at another. There are some more oriented 4-manifolds admitting co-dimension 2 principal orbits. An example is $P(F)$ for F a 2-dimensional surface for which at each boundary circle the principal S^2 orbit degenerates to $\mathbb{R}P^2$. If F is a two 2-disk B^2 , then $P(F) = M(\mathbb{R}P^2)$. For other 2-dimensional surfaces F with boundaries, $P(F)$ gives another class of oriented 4-dimensional manifolds which exhibit co-dimension 2 principal orbits. However, we have not been able to establish whether $P(F)$ are spin for $F \neq B^2$.

Note that in IIB, the Killing spinors have a Spin_c structure rather than a Spin structure. In particular the spinors are twisted with an additional $U(1)$ bundle which is the pull back on the spacetime of canonical $U(1)$ bundle that arises after viewing the target space of IIB scalars as a $SU(1,1)/U(1)$ coset space. As this coset space is the upper half plane, it is contractible unless one considers further identifications which we shall not investigate here. As a result the $U(1)$ bundle is topologically trivial and so this Spin_c structure is equivalent to a standard Spin structure.

D Further conditions on scalar spinor bi-linears

In this appendix, we shall further investigate some properties of the scalar spinor bilinears, and establish more vanishing conditions for some of these. The scalar bilinears are

$$\langle \sigma_+, \sigma_+ \rangle, \quad \langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle, \quad \langle \sigma_+, \Gamma_{ab} \sigma_+ \rangle, \quad \langle \sigma_+, C * \sigma_+ \rangle, \quad \langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle, \quad (\text{D.1})$$

where $a, b = 1, 2, 3, z$. The last two bilinears are twisted with a $U(1)$. We have already shown that for smooth closed AdS_6 backgrounds $\| \sigma_+ \|$ is constant and $\langle \sigma_+, \tilde{\Gamma}_{(4)} \sigma_+ \rangle = \langle \sigma_+, C * \sigma_+ \rangle = 0$. We shall assume that these three conditions are valid throughout this appendix. So it remains to investigate the bilinears $\langle \sigma_+, \Gamma_{ab} \sigma_+ \rangle$ and $\langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle$.

For the latter scalar bilinear, an application of the gravitino KSE (4.7) reveals that

$$\begin{aligned} \nabla_i \langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle &= -\partial_i \log A \langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle - iQ_i \langle \sigma_+, \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle \\ &\quad + \frac{1}{8} \bar{W}_i \| \sigma_+ \|^2 + \frac{3}{8} \bar{W}_m \langle \sigma_+, \Gamma^m_i \sigma_+ \rangle, \end{aligned} \quad (\text{D.2})$$

which may be used to determine the flux W in terms of geometry.

For the $\langle \sigma_+, \Gamma_{ab} \sigma_+ \rangle$ scalar bilinear, we have after an application of the gravitino KSE (4.7) that

$$\nabla_i \langle \sigma_+, \Gamma_{ab} \sigma_+ \rangle = -\partial_i \log A \langle \sigma_+, \Gamma_{ab} \sigma_+ \rangle + \frac{3i}{8} \text{Im} \langle \sigma_+, W_j \Gamma_{ab} \Gamma^j_i \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle \quad (\text{D.3})$$

Setting $a = r, b = s$ for $r, s = 1, 2, 3$ and after using the KSE (4.8), we find the identities

$$\begin{aligned} \frac{A^{-1}}{\ell} \langle \sigma_+, \Gamma_{rszi} \sigma_+ \rangle + \partial_j \log A \langle \sigma_+, \Gamma_{rs} \Gamma^j_i \sigma_+ \rangle - \frac{1}{8} \text{Re} \langle \sigma_+, \Gamma_{rs} \Gamma^j_i W_j \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle &= 0 , \\ \partial_i \log A \text{Im} \langle \sigma_+, \Gamma_{rs} \sigma_+ \rangle + \frac{1}{8} \text{Im} \langle \sigma_+, \Gamma_{rs} \Gamma^j_i W_j \tilde{\Gamma}_{(4)} C * \sigma_+ \rangle &= 0 , \end{aligned} \quad (\text{D.4})$$

and

$$\partial_i (A^4 \langle \sigma_+, \Gamma_{rs} \sigma_+ \rangle) = 0 . \quad (\text{D.5})$$

We shall now demonstrate that

$$\langle \sigma_+, \Gamma_{rs} \sigma_+ \rangle = 0 , \quad (\text{D.6})$$

for smooth closed AdS_6 backgrounds. To prove this, take the derivative of (D.4) and use the gravitino KSE (4.7) and field equation for W , to obtain an expression for $\nabla^2 \log A$ which depends on the scalars ξ . Eliminating the ξ terms using the algebraic KSE in (4.7), we find an expression for $\nabla^2 \log A$ which depends both on $|W|^2$ and $W \wedge \bar{W}$. It turns out that the last term can be eliminated using the algebraic KSE for Ξ in (4.8). The resulting expression is

$$[\nabla^2 \log A + 6(\partial \log A)^2 - \frac{1}{8} |W|^2] \langle \sigma_+, \Gamma_{rs} \sigma_+ \rangle = 0 . \quad (\text{D.7})$$

Observe that because of (D.5), $\text{Im} \langle \sigma_+, \Gamma_{rs} \sigma_+ \rangle$ has a definite sign. If $\langle \sigma_+, \Gamma_{rs} \sigma_+ \rangle \neq 0$, an application of the maximum principle reveals that A is constant and $W = 0$. There are no such smooth closed AdS_6 solutions. This establishes that (D.6) must be satisfied on smooth closed AdS_6 backgrounds.

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